

**DIFFERENTIAL BATALIN-VILKOVISKY ALGEBRAS
ARISING FROM TWILLED LIE-RINEHART ALGEBRAS
BANACH CENTER PUBLICATIONS 51 (2000), 87–102**

JOHANNES HUEBSCHMANN

Université des Sciences et Technologies de Lille
UFR de Mathématiques
F-59 655 VILLENEUVE D’ASCQ Cédex/France
Johannes.Huebschmann@univ-lille1.fr

November 29, 1998

ABSTRACT. Twilled $L(\text{ie-})R(\text{inehart-})$ -algebras generalize, in the Lie-Rinehart context, complex structures on smooth manifolds. An almost complex manifold determines an “almost twilled pre-LR algebra”, which is a true twilled LR-algebra iff the almost complex structure is integrable. We characterize twilled LR structures in terms of certain associated differential (bi)graded Lie and $G(\text{erstenhaber-})$ -algebras; in particular the G -algebra arising from an almost complex structure is a $d(\text{ifferential})$ G -algebra iff the almost complex structure is integrable. Such G -algebras, endowed with a generator turning them into a $B(\text{atalin-})V(\text{ilkovisky-})$ -algebra, occur on the B -side of the mirror conjecture. We generalize a result of Koszul to those dG -algebras which arise from twilled LR-algebras. A special case thereof explains the relationship between holomorphic volume forms and exact generators for the corresponding dG -algebra and thus yields in particular a conceptual proof of the Tian-Todorov lemma. We give a differential homological algebra interpretation for twilled LR-algebras and by means of it we elucidate the notion of generator in terms of homological duality for differential graded LR-algebras.

Introduction

In a series of seminal papers [2], [3], [4], Batalin and Vilkovisky studied the quantization of constrained systems and for that purpose introduced certain differential graded algebras which have later been christened *Batalin-Vilkovisky algebras*. Batalin-Vilkovisky algebras have recently become important in string theory and elsewhere, cf. e. g. [1], [11], [15], [20], [21], [26], [32], [38], [42]. String theory leads to the mysterious mirror conjecture. A version thereof involves certain differential Batalin-Vilkovisky algebras arising from a Calabi-Yau manifold. A crucial ingredient

1991 *Mathematics Subject Classification.* primary 17B55, 17B56, 17B65, 17B66, 17B70, 17B71, secondary 32G05, 53C05, 53C15, 81T70.

Key words and phrases. Lie-Rinehart algebra, twilled Lie-Rinehart algebra, Lie bialgebra, Gerstenhaber algebra, Batalin-Vilkovisky algebra, differential graded Lie algebra, mirror conjecture, Calabi-Yau manifold.

is what is referred to in the literature as the *Tian-Todorov* lemma. This Lemma, in turn, implies the

Here we will give a leisurely introduction to a thorough study of such differential Batalin-Vilkovisky algebras and generalizations thereof in the framework of Lie-Rinehart algebras, thereby trying to avoid technicalities; these and more details may be found in [18].

A *Gerstenhaber* algebra is a graded commutative algebra together with a bracket which (i) yields an ordinary Lie bracket once the underlying module (or vector space) has been regraded down by 1 and which (ii) satisfies a certain derivation property. Such a bracket occurs in Gerstenhaber’s paper [9]. See Section 2 below for details. A differential Batalin-Vilkovisky algebra is a Gerstenhaber algebra together with an exact generator, and the underlying Gerstenhaber algebras of interest for us, in turn, arise as (bigraded) algebras of forms on *twilled Lie-Rinehart algebras* (which we introduce below). In the Lie-Rinehart context, a twilled Lie-Rinehart algebra generalizes, among others, the notion of a complex structure on a smooth manifold. One of our results, Theorem 2.3 below, says that an “almost twilled Lie-Rinehart algebra” is a true twilled Lie-Rinehart algebra if and only if the corresponding Gerstenhaber algebra is a differential Gerstenhaber algebra. This implies, for example, that the integrability condition for an almost complex structure on a smooth manifold may be phrased as a condition saying that a certain operator on the corresponding Gerstenhaber algebra turns the latter into a differential Gerstenhaber algebra. Now a theorem of Koszul [25] establishes, on an ordinary smooth manifold, a bijective correspondence between generators for the Gerstenhaber algebra of multi vector fields and connections in the top exterior power of the tangent bundle in such a way that exact generators correspond to flat connections. In Theorem 2.7 below we generalize this bijective correspondence to the differential Gerstenhaber algebras arising from twilled Lie-Rinehart algebras; such Gerstenhaber algebras come into play, for example, in the mirror conjecture. What corresponds to a flat connection on the line bundle in Koszul’s theorem is now a holomorphic volume form—its existence is implied by the Calabi-Yau condition—and our generalization of Koszul’s theorem shows in particular how a holomorphic volume form determines a generator for the corresponding differential Gerstenhaber algebra turning it into a differential Batalin-Vilkovisky algebra. The resulting differential Batalin-Vilkovisky algebra then generalizes that which underlies what is called the B-model. In particular, as a consequence of our methods, we obtain a new proof of the Tian-Todorov lemma. We also give a differential homological algebra interpretation of twilled Lie-Rinehart algebras and, furthermore, of a generator for a differential Batalin-Vilkovisky algebra in terms of a suitable notion of homological duality. This relies on results in our earlier papers [14] and [15] as well as on various generalizations thereof.

I am indebted to Y. Kosmann-Schwarzbach and K. Mackenzie for discussions, and to J. Stasheff and A. Weinstein for some e-mail correspondence about various topics related with the paper. At the “Poissonfest”, Y. Kosmann-Schwarzbach introduced me to the recent manuscript [36] which treats topics somewhat related to the present paper. There is little overlap, though.

It is a pleasure to thank the organizers of the “Poissonfest” for the opportunity to present these results. The material to be presented here is related to some of

the work of the late S. Zakrzewski; see Remark 4.3 below. We respectfully dedicate this paper to his memory.

1. Twilled Lie-Rinehart algebras

Let R be a commutative ring. A *Lie-Rinehart algebra* (A, L) consists of a commutative R -algebra A and an R -Lie algebra L together with an A -module structure $A \otimes_R L \rightarrow L$ on L , written $a \otimes_R \alpha \mapsto a\alpha$, and an action $L \rightarrow \text{Der}(A)$ of L on A (which is a morphism of R -Lie algebras and) whose adjoint $L \otimes_R A \rightarrow A$ is written $\alpha \otimes_R a \mapsto \alpha(a)$; here $a \in A$ and $\alpha \in L$. These mutual actions are required to satisfy certain compatibility properties modeled on $(A, L) = (C^\infty(M), \text{Vect}(M))$ where $C^\infty(M)$ and $\text{Vect}(M)$ refer to the algebra of smooth functions and the Lie algebra of smooth vector fields, respectively, on a smooth manifold M . In general, the compatibility conditions read:

$$(1.1) \quad (a\alpha)b = a\alpha(b), \quad a, b \in A, \quad \alpha \in L,$$

$$(1.2) \quad [\alpha, a\beta] = \alpha(a)\beta + a[\alpha, \beta], \quad a \in A, \quad \alpha, \beta \in L.$$

For a Lie-Rinehart algebra (A, L) , following [35], we will refer to L as an (R, A) -*Lie algebra*. In differential geometry, (R, A) -Lie algebras arise as spaces of sections of Lie algebroids. Lie-Rinehart algebras have been studied before Rinehart by Herz [12] under the name “pseudo-algèbre de Lie” as well as by Palais [34] under the name “ d -Lie-ring”. We have chosen to refer to these object as Lie-Rinehart algebras since Rinehart subsumed their cohomology under standard homological algebra and established a Poincaré-Birkhoff-Witt theorem for them [35]. In particular, Rinehart has shown how to describe de Rham cohomology in the language of Ext-groups. In a sense, the homological algebra interpretations of differential Batalin-Vilkovisky algebras to be given below push these observations of Rinehart’s further.

Given two Lie-Rinehart algebras (A, L') and (A, L'') , together with mutual actions $\cdot: L' \otimes_R L'' \rightarrow L''$ and $\cdot: L'' \otimes_R L' \rightarrow L'$ which endow L'' and L' with an (A, L') - and (A, L'') -module structure, respectively, we will refer to (A, L', L'') as an *almost twilled Lie-Rinehart algebra*; we will call it a *twilled Lie-Rinehart algebra* provided the direct sum A -module structure on $L = L' \oplus L''$, the sum $(L' \oplus L'') \otimes_R A \rightarrow A$ of the adjoints of the L' - and L'' -actions on A , and the bracket $[\cdot, \cdot]$ on $L = L' \oplus L''$ given by

$$(1.3) \quad [(\alpha'', \alpha'), (\beta'', \beta')] = [\alpha'', \beta''] + [\alpha', \beta'] + \alpha'' \cdot \beta' - \beta' \cdot \alpha'' + \alpha' \cdot \beta'' - \beta'' \cdot \alpha'$$

turn (A, L) into a Lie-Rinehart algebra. We then write $L = L' \bowtie L''$ and refer to (A, L) as the *twilled sum* of (A, L') and (A, L'') .

For illustration, consider a smooth manifold M with an almost complex structure, let A be the algebra of smooth complex functions on M , L the (\mathbb{C}, A) -Lie algebra of complexified smooth vector fields on M , and consider the ordinary decomposition of the complexified tangent bundle $\tau_M^{\mathbb{C}}$ as a direct sum $\tau_M' \oplus \tau_M''$ of the *almost holomorphic* and *almost antiholomorphic* tangent bundles τ_M' and τ_M'' , respectively; write L' and L'' for their spaces of smooth sections. Then (A, L', L'') , together with the mutual actions coming from L , is a twilled Lie-Rinehart algebra if and only if the almost complex structure is integrable, i. e. a true complex structure; τ_M' and τ_M'' are then the ordinary *holomorphic* and *antiholomorphic* tangent bundles,

respectively. The precise analogue of an almost complex structure is what we call an *almost twilled pre-Lie-Rinehart algebra* structure; this notion is weaker than that of almost twilled Lie-Rinehart algebra. The basic difference is that, for an almost twilled pre-Lie-Rinehart algebra, instead of having mutual actions $\cdot : L' \otimes_R L'' \rightarrow L''$ and $\cdot : L'' \otimes_R L' \rightarrow L'$, we only require that there be given R -linear pairings $\cdot : L' \otimes_R L'' \rightarrow L''$ and $\cdot : L'' \otimes_R L' \rightarrow L'$, which endow L'' and L' with an (A, L') - and (A, L'') -connection, respectively; see [18] for details. A situation similar to that of a complex structure on a smooth manifold and giving rise to a twilled Lie-Rinehart algebra arises from a smooth manifold with two transverse foliations as well as from a Cauchy-Riemann structure (cf. [7]); see [18] for some comments about Cauchy-Riemann structures. Lie bialgebras provide another class of examples of twilled Lie-Rinehart algebras; Kosmann-Schwarzbach and F. Magri refer to these objects, or rather to the corresponding twilled sum, as *twilled extensions of Lie algebras* [24]; Lu and Weinstein call them *double Lie algebras* [27]; and Majid uses the terminology *matched pairs* of Lie algebras [31]. Spaces of sections of suitable pairs of Lie algebroids with additional structure lead to yet another class of examples of twilled Lie-Rinehart algebras; these have been studied in the literature under the name *matched pairs of Lie algebroids* by Mackenzie [28] and Mokri [33].

An almost twilled Lie-Rinehart algebra (A, L'', L') is a true twilled Lie-Rinehart algebra if and only if (A, L'', L') satisfies three compatibility conditions; these are spelled out in [18] (Proposition 1.7). This proposition is merely an adaption of earlier results in the literature to our more general situation. Another interpretation of the compatibility conditions involves certain annihilation properties of the two operators d' and d'' which arise, given an almost twilled pre-Lie-Rinehart algebra (A, L', L'') , as formal extensions of the ordinary Lie-Rinehart differentials with respect to L' and L'' , respectively, on the bigraded algebra $\text{Alt}_A^*(L'', \text{Alt}_A^*(L', A))$ (but are not necessarily exact): To explain this, we will say that an A -module M has *property P* provided for $x \in M$, $\phi(x) = 0$ for every $\phi : M \rightarrow A$ implies that x is zero. For example, a projective A -module has property P. We now have the following, cf. Theorem 1.15 in [18].

Theorem 1.4. *If (A, L', L'') is a twilled Lie-Rinehart algebra, the operators d' and d'' turn the bigraded algebra $\text{Alt}_A^*(L'', \text{Alt}_A^*(L', A))$ into a differential bigraded algebra which then necessarily computes the cohomology $H^*(\text{Alt}_A(L, A))$ of the twilled sum L of L' and L'' . Conversely, given an almost twilled pre-Lie-Rinehart algebra (A, L', L'') , if the operators d' and d'' turn the bigraded algebra $\text{Alt}_A^*(L'', \text{Alt}_A^*(L', A))$ into a differential bigraded algebra and if L' and L'' have property P, (A, L', L'') is a true twilled Lie-Rinehart algebra.*

For example, for the twilled Lie-Rinehart algebra arising from the holomorphic and antiholomorphic tangent bundles of a complex manifold, the resulting differential bigraded algebra $(\text{Alt}_A^*(L'', \text{Alt}_A^*(L', A)), d', d'')$ comes down to the ordinary Dolbeault complex.

2. Twilled Lie-Rinehart algebras, Gerstenhaber algebras, and dBV-algebras

We now explain briefly how other characterizations of twilled Lie-Rinehart algebras explain the structure of certain differential Batalin-Vilkovisky algebras. Section 2 of [18] is devoted to more details about differential graded Lie-Rinehart algebras.

Let (A, L) be a Lie-Rinehart algebra, and let \mathcal{A} be a graded commutative A -algebra which is endowed with a graded (A, L) -module structure in such a way that (i) L acts on \mathcal{A} by derivations—this is equivalent to requiring the structure map from $\mathcal{A} \otimes_A \mathcal{A}$ to \mathcal{A} to be a morphism of graded (A, L) -modules—and that (ii) the canonical map from A to \mathcal{A} is a morphism of left (A, L) -modules. Let $\mathcal{L} = \mathcal{A} \otimes_A L$, and define a bigraded bracket

$$(2.1.1) \quad [\cdot, \cdot]: \mathcal{L} \otimes_R \mathcal{L} \rightarrow \mathcal{L}$$

of bidegree $(0, -1)$ by means of the formula

$$(2.1.2) \quad [\alpha \otimes_A x, \beta \otimes_A y] = (\alpha\beta) \otimes_A [x, y] + \alpha(x \cdot \beta) \otimes_A y - (-1)^{|\alpha||\beta|} \beta(y \cdot \alpha) \otimes_A x$$

where $\alpha, \beta \in \mathcal{A}$ and $x, y \in L$. A calculation shows that, for every $\beta \in \mathcal{A}$ and every $x, y, z \in L$,

$$[[x, y], \beta \otimes_A z] - ([x, [y, \beta \otimes_A z]] - [y, [x, \beta \otimes_A z]]) = ([x, y](\beta) - x(y(\beta)) - y(x(\beta))) \otimes_A z,$$

whence (2.1.1) being a graded Lie bracket is actually equivalent to the structure map $L \otimes_R \mathcal{A} \rightarrow \mathcal{A}$ being a Lie algebra action. Moreover, let

$$(2.1.3) \quad \mathcal{A} \otimes_R \mathcal{L} \rightarrow \mathcal{L}$$

be the obvious graded left \mathcal{A} -module structure arising from extension of scalars, that is from extending L to a (graded) \mathcal{A} -module, and define a pairing

$$(2.1.4) \quad \mathcal{L} \otimes_R \mathcal{A} \rightarrow \mathcal{A}$$

by

$$(2.1.5) \quad (\alpha \otimes_A x) \otimes_R \beta \mapsto (\alpha \otimes_A x)(\beta) = \alpha(x(\beta)).$$

Then $(\mathcal{A}, \mathcal{L})$, together with (2.1.1), (2.1.3) and (2.1.4), constitutes a graded Lie-Rinehart algebra. We refer to $(\mathcal{A}, \mathcal{L})$ as the *crossed product* of \mathcal{A} and (A, L) and to the corresponding (R, \mathcal{A}) -Lie algebra \mathcal{L} as the *crossed product* of \mathcal{A} and L . More details about this notion of graded crossed product Lie-Rinehart algebra may be found in [18] (2.8).

REMARK 2.1.6. We must be a little circumspect here: The three terms on the right-hand side of (2.1.2) are *not* well defined individually; only their sum is well defined. For example, if we take ax instead of x , where $a \in A$, on the left-hand side, $\alpha \otimes_A (ax)$ equals $(\alpha a) \otimes_A x$ but $(\alpha\beta) \otimes_A [ax, y]$ differs from $(\alpha a\beta) \otimes_A [x, y]$.

Let (A, L'', L') be an almost twilled Lie-Rinehart algebra having L' finitely generated and projective as an A -module. Write $\mathcal{A}'' = \text{Alt}_A(L'', A)$ and $\mathcal{L}' = \text{Alt}_A(L'', L')$. Now \mathcal{A}'' is a graded commutative A -algebra and, endowed with the Lie-Rinehart differential d'' (which corresponds to the (R, A) -Lie algebra structure on L''), \mathcal{A}'' is a differential graded commutative R -algebra. Moreover, from the (A, L'') -module structure on L' , \mathcal{L}' inherits an obvious differential graded \mathcal{A}'' -module structure. Furthermore, the (A, L') -structure on L'' induces an action of L' on \mathcal{A}''

by graded derivations. Since L' is supposed to be finitely generated and projective as an A -module, the canonical A -module morphism

$$\mathcal{A}'' \otimes_A L \rightarrow \mathcal{L}' = \text{Alt}_A(L'', L')$$

is an isomorphism of graded A -modules, in fact of graded \mathcal{A}'' -modules. Taking $L = L'$ and $\mathcal{A} = \mathcal{A}''$, the graded crossed product Lie-Rinehart structure explained above is available, and we have the graded crossed product Lie-Rinehart algebra $(\mathcal{A}'', \mathcal{L}')$. Now the (R, A) -Lie algebra structure on L'' and the (A, L'') -module structure on L' determine the corresponding Lie-Rinehart differential on $\mathcal{L}' = \text{Alt}_A(L'', L')$; we denote it by d'' . By symmetry, when L'' is finitely generated and projective as an A -module, we have the same structure, with L' and L'' interchanged.

Theorem 2.1. *As an A -module, L' being supposed to be finitely generated and projective, the statements (i), (ii), and (iii) below are equivalent:*

- (i) (A, L'', L') is a true twilled Lie-Rinehart algebra;
- (ii) $(\mathcal{L}', d'') = (\text{Alt}_A(L'', L'), d'')$ is a differential graded R -Lie algebra;
- (iii) $(\mathcal{A}'', \mathcal{L}'; d'')$ is a differential graded Lie-Rinehart algebra.

Thus, under these circumstances, there is a bijective correspondence between twilled Lie-Rinehart algebra and differential graded Lie-Rinehart algebra structures.

For a proof of this result and for more details, see (3.2) in [18]. We note that, in the situation of Theorem 2.1, the Lie bracket on $\mathcal{L}' = \text{Alt}_A(L'', L')$ does not just come down to the shuffle product of forms on L'' and the Lie bracket on L' ; in fact, such a bracket would not even be well defined since the Lie bracket of L' is not A -linear, i. e., in the usual differential geometry context, does not behave as a “tensor”.

In [17], a notion of graded crossed product Lie-Rinehart algebra has been given which may entirely be described in terms of the A -modules of forms $\text{Alt}_A(L'', A)$ and $\text{Alt}_A(L'', L')$; by means thereof, the statement of Theorem 2.1 may be established without the hypothesis that L' be finitely generated and projective as an A -module.

When (A, L', L'') is the twilled Lie-Rinehart algebra arising from the holomorphic and antiholomorphic tangent bundles of a smooth complex manifold M , $(\mathcal{L}', d'') = (\text{Alt}_A(L'', L'), d'')$ is what is called the *Kodaira-Spencer* algebra in the literature; it controls the infinitesimal deformations of the complex structure on M . The cohomology $H^*(L'', L')$ then inherits a graded Lie algebra structure and the obstruction to deforming the complex structure is the map $H^1(L'', L') \rightarrow H^2(L'', L')$ which sends $\eta \in H^1(L'', L')$ to $[\eta, \eta] \in H^2(L'', L')$.

Recall that a *Gerstenhaber algebra* is a graded commutative R -algebra \mathcal{A} together with a graded Lie bracket from $\mathcal{A} \otimes_R \mathcal{A}$ to \mathcal{A} of degree -1 (in the sense that, if \mathcal{A} is regraded down by one, $[\cdot, \cdot]$ is an ordinary graded Lie bracket) such that, for each homogeneous element a of \mathcal{A} , $[a, \cdot]$ is a derivation of \mathcal{A} of degree $|a| - 1$ where $|a|$ refers to the degree of a ; see [10] where these objects are called G-algebras, or [15, 21, 26, 42]. In our paper [15], we worked out an intimate link between Gerstenhaber’s and Rinehart’s papers [9] and [35] which involves the notion of Gerstenhaber bracket. In a sense, in the present paper we extend this link to the differential graded situation.

Given a bigraded commutative R -algebra \mathcal{A} , we will say that a bigraded bracket $[\cdot, \cdot]: \mathcal{A} \otimes_R \mathcal{A} \rightarrow \mathcal{A}$ of bidegree $(0, -1)$ is a *bigraded Gerstenhaber bracket* provided

$[\cdot, \cdot]$ is an ordinary bigraded Lie bracket when the second degree of \mathcal{A} is regraded down by one, the first one being kept, such that, for each homogeneous element a of \mathcal{A} of bidegree (p, q) , $[a, \cdot]$ is a derivation of \mathcal{A} of bidegree $(p, q - 1)$; a bigraded R -algebra with a bigraded Gerstenhaber bracket will be referred to as a *bigraded Gerstenhaber algebra*. Moreover, given a bigraded Gerstenhaber algebra $(\mathcal{A}, [\cdot, \cdot])$ together with a differential d of bidegree $(1, 0)$ which endows \mathcal{A} with a differential graded R -algebra structure we will say that $(\mathcal{A}, [\cdot, \cdot])$ and d constitute a *differential bigraded Gerstenhaber algebra* (or differential bigraded G-algebra), written $(\mathcal{A}, [\cdot, \cdot], d)$, provided d behaves as a derivation for the bigraded Gerstenhaber bracket $[\cdot, \cdot]$, that is,

$$d[x, y] = [dx, y] - (-1)^{|x|}[x, dy], \quad x, y \in \mathcal{A},$$

where the total degree $|x|$ is the sum of the bidegrees.

Recall that, given a Lie-Rinehart algebra (A, L) , the Lie bracket on L determines a Gerstenhaber bracket on the exterior A -algebra $\Lambda_A L$ on L ; for $\alpha_1, \dots, \alpha_n \in L$, the bracket $[u, v]$ in $\Lambda_A L$ of $u = \alpha_1 \wedge \dots \wedge \alpha_\ell$ and $v = \alpha_{\ell+1} \wedge \dots \wedge \widehat{\alpha_n}$ is given by the expression

$$(2.2.1) \quad [u, v] = (-1)^\ell \sum_{j \leq \ell < k} (-1)^{(j+k)} [\alpha_j, \alpha_k] \wedge \alpha_1 \wedge \dots \wedge \widehat{\alpha_j} \wedge \dots \wedge \widehat{\alpha_k} \wedge \dots \wedge \alpha_n,$$

where $\ell = |u|$ is the degree of u , cf. [15] (1.1).

We now return to a general almost twilled Lie-Rinehart algebra (A, L', L'') having L' finitely generated and projective as an A -module and consider the graded crossed product Lie-Rinehart algebra $(\mathcal{A}'', \mathcal{L}')$. The graded Lie-Rinehart bracket on $\mathcal{L}' (= \text{Alt}_A(L'', L'))$ extends to a (bigraded) bracket on $\text{Alt}_A(L'', \Lambda_A L')$ which turns the latter into a bigraded Gerstenhaber algebra; as a bigraded algebra, $\text{Alt}_A(L'', \Lambda_A L')$ could be thought as of the exterior \mathcal{A}'' -algebra on \mathcal{L}' , and we write sometimes

$$\Lambda_{\mathcal{A}''} \mathcal{L}' = \text{Alt}_A(L'', \Lambda_A L').$$

With reference to the graded Lie bracket $[\cdot, \cdot]$ on \mathcal{L}' and the L' -action on \mathcal{A}'' , the bigraded Gerstenhaber bracket

$$(2.2.2) \quad [\cdot, \cdot]: \Lambda_{\mathcal{A}''} \mathcal{L}' \otimes_R \Lambda_{\mathcal{A}''} \mathcal{L}' \rightarrow \Lambda_{\mathcal{A}''} \mathcal{L}'$$

on $\Lambda_{\mathcal{A}''} \mathcal{L}'$ may be described by the formulas

$$(2.2.3) \quad \begin{aligned} [\alpha\beta, \gamma] &= \alpha[\beta, \gamma] + (-1)^{|\alpha||\beta|} \beta[\alpha, \gamma], \quad \alpha, \beta, \gamma \in \Lambda_{\mathcal{A}''} \mathcal{L}', \\ [x, a] &= x(a), \quad x \in L', a \in \mathcal{A}'', \\ [\alpha, \beta] &= -(-1)^{(|\alpha|-1)(|\beta|-1)} [\beta, \alpha], \quad \alpha, \beta \in \Lambda_{\mathcal{A}''} \mathcal{L}', \end{aligned}$$

where $|\cdot|$ refers to the total degree, i.e. the sum of the two bidegree components. The bracket (2.2.2) is in fact the *(bigraded) crossed product bracket extension* of the Gerstenhaber bracket on $\Lambda_A L'$ and $\Lambda_{\mathcal{A}''} \mathcal{L}'$ may be viewed as the *(bigraded) crossed product Gerstenhaber algebra* of \mathcal{A}'' with the ordinary Gerstenhaber algebra $\Lambda_A L'$. See Section 4 of [18] for details.

The Lie-Rinehart differential d'' which corresponds to the Lie-Rinehart structure on L'' and the induced graded (A, L'') -module structure on $\Lambda_A L'$ turn $\text{Alt}_A(L'', \Lambda_A L')$ into a differential (bi)-graded commutative R -algebra. By symmetry, when L'' is finitely generated and projective as an A -module, we have the same structure, with L' and L'' interchanged.

Theorem 2.3. *The almost twilled Lie-Rinehart algebra (A, L'', L') is a true twilled Lie-Rinehart algebra if and only if $(\Lambda_{A''} L', d'')$ $(= (\text{Alt}_A(L'', \Lambda_A L'), d''))$ is a differential (bi)-graded Gerstenhaber algebra.*

See Theorem 4.4 in [18] and its proof.

When (A, L', L'') arises from the holomorphic and antiholomorphic tangent bundles of a smooth complex manifold M , the resulting differential Gerstenhaber algebra $(\text{Alt}_A(L'', \Lambda_A L'), d'')$ is that of forms of type $(0, *)$ with values in the holomorphic multi vector fields, the operator d'' being the Cauchy-Riemann operator (which is more usually written $\bar{\partial}$). This differential Gerstenhaber algebra comes into play in the mirror conjecture; it was studied by Barannikov-Kontsevich [1], Manin [32], Witten [41], and others.

Let now (A, L'', L') be a twilled Lie-Rinehart algebra having L' finitely generated and projective as an A -module of constant rank n (say), and write $\Lambda_A^n L'$ for the top exterior power of L' over A . Consider the differential Gerstenhaber algebra $(\text{Alt}_A(L'', \Lambda_A L'), d'')$. Our next aim is to study generators thereof. To this end, we observe that, when $\text{Alt}_A(L', \Lambda_A^n L')$ is endowed with the obvious graded (A, L'') -module structure induced from the left (A, L'') -module structure on L' which is part of the structure of twilled Lie-Rinehart algebra, the canonical isomorphism

$$(2.4) \quad \text{Alt}_A(L'', \Lambda_A L') \rightarrow \text{Alt}_A(L'', \text{Alt}_A(L', \Lambda_A^n L'))$$

of graded A -modules is compatible with the differentials which correspond to the Lie-Rinehart structure on L'' and the (A, L'') -module structures on the coefficients on both sides of (2.4); abusing notation, we denote each of these differentials by d'' .

For a bigraded Gerstenhaber algebra \mathcal{A} over R , with bracket operation written $[\cdot, \cdot]$, an R -linear operator Δ on \mathcal{A} of bidegree $(0, -1)$ will be said to *generate* the Gerstenhaber bracket provided, for every homogeneous $a, b \in \mathcal{A}$,

$$(2.5) \quad [a, b] = (-1)^{|a|} \left(\Delta(ab) - (\Delta a)b - (-1)^{|a|} a(\Delta b) \right);$$

the operator Δ is then called a *generator*. A generator Δ is said to be *exact* provided $\Delta\Delta$ is zero, that is, Δ is a differential; an exact generator will henceforth be written ∂ . A bigraded Gerstenhaber algebra \mathcal{A} together with a generator Δ will be called a *weak bigraded Batalin-Vilkovisky algebra* (or weak bigraded BV-algebra); when the generator is exact, written ∂ , we will refer to (\mathcal{A}, ∂) (more simply) as a *bigraded Batalin-Vilkovisky algebra* (or bigraded BV-algebra).

It is clear that a generator determines the bigraded Gerstenhaber bracket. An observation due to Koszul [25] (p. 261) carries over to the bigraded case: for any bigraded Batalin-Vilkovisky algebra $(\mathcal{A}, [\cdot, \cdot], \partial)$, the operator ∂ (which is exact by assumption) behaves as a derivation for the bigraded Gerstenhaber bracket $[\cdot, \cdot]$, that is,

$$(2.6) \quad \partial[x, y] = [\partial x, y] - (-1)^{|x|} [x, \partial y], \quad x, y \in \mathcal{A}.$$

An exact generator ∂ does in general *not* behave as a derivation for the multiplication of \mathcal{A} , though. Let (\mathcal{A}, Δ) be a weak bigraded Batalin-Vilkovisky algebra, write $[\cdot, \cdot]$

for the bigraded Gerstenhaber bracket generated by Δ , and let d be a differential of bidegree $(+1, 0)$ which endows $(\mathcal{A}, [\cdot, \cdot])$ with a differential bigraded Gerstenhaber algebra structure. Consider the graded commutator $[d, \Delta] = d\Delta + \Delta d$ on \mathcal{A} ; it is an operator of bidegree $(1, -1)$ and hence of total degree zero. We will say that (\mathcal{A}, Δ, d) is a *weak differential* bigraded Batalin-Vilkovisky algebra provided the commutator $[d, \Delta]$ is zero. In particular, a weak differential bigraded Batalin-Vilkovisky algebra $(\mathcal{A}, \partial, d)$ which has ∂ exact will be called a *differential bigraded Batalin-Vilkovisky algebra*. On the underlying bigraded object \mathcal{A} of a differential bigraded Batalin-Vilkovisky algebra $(\mathcal{A}, \partial, d)$, the graded commutator $[d, \partial]$ manifestly behaves as a derivation for the bigraded Gerstenhaber bracket since d and ∂ both behave as derivations for this bracket.

We now reproduce the statement of Theorem 5.4.6 in [18].

Theorem 2.7. *The isomorphism (2.4) furnishes a bijective correspondence between generators of the bigraded Gerstenhaber structure on the left-hand side of (2.4) and (A, L') -connections on $\Lambda_A^n L'$ in such a way that exact generators correspond to (A, L') -module structures (i. e. flat connections). Under this correspondence, generators of the differential bigraded Gerstenhaber structure on the left-hand side correspond to (A, L') -connections on $\Lambda_A^n L'$ which are compatible with the (A, L'') -module structure on $\Lambda_A^n L'$.*

Thus, in particular, exact generators of the differential bigraded Gerstenhaber structure on the left-hand side correspond to (A, L'') -compatible (A, L') -module structures on $\Lambda_A^n L'$.

When L'' is trivial and L' the Lie algebra of smooth vector fields on a smooth manifold, the statement of this theorem comes down to the result of Koszul [25] mentioned earlier. Our result not only provides many examples of differential Batalin-Vilkovisky algebras but also explains how every differential Batalin-Vilkovisky algebra having an underlying bigraded A -algebra of the kind $\text{Alt}_A(L'', \Lambda_A L')$ arises.

When (A, L', L'') is the twilled Lie-Rinehart algebra which comes from the holomorphic and antiholomorphic tangent bundles of a smooth complex manifold M as explained above, the theorem gives a bijective correspondence between generators of the differential bigraded Gerstenhaber algebra $(\text{Alt}_A(L'', \Lambda_A L'), d'')$ of forms of type $(0, *)$ with values in the holomorphic multi vector fields, the differential d'' being the Cauchy-Riemann operator, and holomorphic connections on the highest exterior power of the holomorphic tangent bundle in such a way that exact generators correspond to flat holomorphic connections. In particular, suppose that M is a *Calabi-Yau* manifold, that is, admits a holomorphic volume form Ω (say). This holomorphic volume form identifies the highest exterior power of the holomorphic tangent bundle with the algebra of smooth complex functions on M as a module over $L = L'' \oplus L'$, hence induces a flat holomorphic connection thereupon and thence an exact generator ∂_Ω for $(\text{Alt}_A(L'', \Lambda_A L'), d'')$, turning the latter into a differential (bi)graded Batalin-Vilkovisky algebra. This is precisely the differential (bi)graded Batalin-Vilkovisky algebra coming into play on the B-side of the mirror conjecture and studied in the cited sources. The fact that the holomorphic volume form induces a generator for the differential Gerstenhaber structure is referred to in the literature as the *Tian-Todorov lemma*; cf [39], [40]. In our approach, this lemma drops out as a special case of our generalization of Koszul's theorem to the bigraded setting,

and this generalization indeed provides a conceptual proof of the lemma. This lemma implies that, for a Kählerian Calabi-Yau manifold M , the deformations of the complex structure are unobstructed, that is to say, there is an open subset of $H^1(M, \tau_M)$ parametrizing the deformations of the complex structure; here $H^1(M, \tau_M)$ is the first cohomology group of M with values in the holomorphic tangent bundle τ_M . Under these circumstances, after a choice of holomorphic volume form Ω has been made, the canonical isomorphism (2.4), combined with the isomorphism

$$\Omega^\flat: \text{Alt}_A(L'', \text{Alt}_A(L', \Lambda_A^n L')) \rightarrow \text{Alt}_A(L'', \text{Alt}_A(L', A))$$

induced by Ω identifies $(\text{Alt}_A(L'', \Lambda_A L'), d'', \partial_\Omega)$ with the Dolbeault complex of M and hence the cohomology $H^*(\text{Alt}_A(L'', \Lambda_A L'), d'', \partial_\Omega)$ with the ordinary complex valued cohomology of M . This is nowadays well understood. The cohomology $H^*(\text{Alt}_A(L'', \Lambda_A L'), d'', \partial_\Omega)$ is referred to in the literature as the *extended moduli space of complex structures* [41]; it underlies what is called the *B-model* in the theory of mirror symmetry.

3. Twilled Lie-Rinehart algebras and differential homological algebra

We now spell out interpretations of some of the above results in terms of differential homological algebra.

Let (A, L', L'') be a twilled Lie-Rinehart algebra having L' and L'' finitely generated and projective as A -modules. Let $(\mathcal{A}'', \mathcal{L}'; d'')$ be the differential graded crossed product Lie-Rinehart algebra $(\text{Alt}_A(L'', A), \text{Alt}_A(L'', L'); d'')$ mentioned before. Let $L = L' \bowtie L''$ be the twilled sum of L' and L'' , and consider the differential graded Lie-Rinehart cohomology $H^*(\mathcal{L}', \mathcal{A}'')$. see Section 6 in [18] for details where also a proof of the following result may be found; cf. in particular Theorem 6.15 in [18].

Theorem 3.1. *The differential bigraded algebra $(\text{Alt}_A^*(L'', \text{Alt}_A^*(L', A)), d', d'')$ computes the differential graded Lie-Rinehart cohomology $H^*(\mathcal{L}', \mathcal{A}'')$. Moreover, this differential graded Lie-Rinehart cohomology is naturally isomorphic to the Lie-Rinehart cohomology $H^*(L, A)$.*

When L'' is trivial, so that $H^*(\mathcal{L}', \mathcal{A}'')$ is an ordinary (ungraded) Lie-Rinehart algebra (A, L) , the differential graded Lie-Rinehart cohomology boils down to the ordinary Lie-Rinehart cohomology $H^*(L, A)$. Moreover, for the special case when A and L are the algebra of smooth functions and smooth vector fields on a smooth manifold, the Lie-Rinehart cohomology $H^*(L, A)$ amounts to the de Rham cohomology; this fact has been established by Rinehart [35]. In our more general situation, when the twilled Lie-Rinehart algebra (A, L', L'') arises from the holomorphic and anti-holomorphic tangent bundles of a smooth complex manifold, the complex calculating the differential graded Lie-Rinehart cohomology $H^*(\mathcal{L}', \mathcal{A}'')$ of the differential graded crossed product Lie-Rinehart algebra $(\mathcal{A}'', \mathcal{L}'; d'') = (\text{Alt}_A(L'', A), \text{Alt}_A(L'', L'); d'')$ is the Dolbeault complex, and the differential graded Lie-Rinehart cohomology amounts to the Dolbeault cohomology. Thus our approach provides, in particular, an interpretation of the Dolbeault complex in the framework of differential homological algebra.

Generalizing results in our earlier paper [15], we can now elucidate the concept of generator of a differential bigraded Batalin-Vilkovisky algebra in the framework of

homological duality for differential graded Lie-Rinehart algebras in the following way: *An exact generator amounts to the differential in a standard complex computing differential graded Lie-Rinehart homology (!) with appropriate coefficients*; see Proposition 7.13 in [18] for details. It may then be shown that, when the appropriate additional structure (in terms of Lie-Rinehart differentials and dBV-generators) is taken into account, the above isomorphism (2.4) is essentially just a duality isomorphism in the (co)homology of the differential graded crossed product Lie-Rinehart algebra $(\mathcal{A}'', \mathcal{L}')$; see Proposition 7.14 in [18] for details. In particular, the Tian-Todorov Lemma comes down to a statement about differential graded (co)homological duality.

4. Twilled Lie-Rinehart algebras and Lie-Rinehart bialgebras

Twilled Lie-Rinehart algebras thus generalize Lie bialgebras, and the twilled sum is an analogue, even a generalization, of the Manin double of a Lie bialgebra. The Lie bialgebroids introduced by Mackenzie and Xu [29] generalize Lie bialgebras as well, and there is a corresponding notion of Lie-Rinehart bialgebra. However, twilled Lie-Rinehart algebras and Lie-Rinehart bialgebras are different, in fact non-equivalent notions which both generalize Lie bialgebras. In a sense, Lie-Rinehart bialgebras generalize Poisson and in particular symplectic structures while twilled Lie-Rinehart algebras generalize complex structures. We now give a characterization of twilled Lie-Rinehart algebras in terms of Lie-Rinehart bialgebras. See Theorem 4.8 in [18] for more details. Let L and D be (R, A) -Lie algebras which, as A -modules, are finitely generated and projective, in such a way that, as an A -module, D is isomorphic to $L^* = \text{Hom}_A(L, A)$. We say that L and D are *in duality*. We write d for the differential on $\text{Alt}_A(L, A) \cong \Lambda_A D$ coming from the Lie-Rinehart structure on L and d_* for the differential on $\text{Alt}_A(D, A) \cong \Lambda_A L$ coming from the Lie-Rinehart structure on D . Likewise we denote the Gerstenhaber bracket on $\Lambda_A L$ coming from the Lie-Rinehart structure on L by $[\cdot, \cdot]$ and that on $\Lambda_A D$ coming from the Lie-Rinehart structure on D by $[\cdot, \cdot]_*$. We will say that (A, L, D) constitutes a *Lie-Rinehart bialgebra* if the differential d_* on $\text{Alt}_A(D, A) \cong \Lambda_A L$ and the Gerstenhaber bracket $[\cdot, \cdot]$ on $\Lambda_A L$ are related by

$$d_*[x, y] = [d_*x, y] + [x, d_*y], \quad x, y \in L,$$

or equivalently, if the differential d on $\text{Alt}_A(L, A) \cong \Lambda_A D$ behaves as a derivation for the Gerstenhaber bracket $[\cdot, \cdot]_*$ in all degrees, that is to say

$$d[x, y]_* = [dx, y]_* - (-1)^{|x|}[x, dy]_*, \quad x, y \in \Lambda_A D.$$

Thus, for a Lie-Rinehart bialgebra (A, L, D) ,

$$(\Lambda_A L, [\cdot, \cdot], d_*) = (\text{Alt}_A(D, A), [\cdot, \cdot]_*, d_*)$$

is a strict *differential Gerstenhaber algebra*, and the same is true of

$$(\Lambda_A D, [\cdot, \cdot]_*, d) = (\text{Alt}_A(L, A), [\cdot, \cdot], d);$$

see [21] (3.5) for details. In fact, a straightforward extension of an observation of Y. Kosmann-Schwarzbach [21] shows that Lie-Rinehart bialgebra structures on

(A, L, D) and strict differential Gerstenhaber algebra structures on $(\Lambda_A L, [\cdot, \cdot], d_*)$ or, what amounts to the same, on $(\Lambda_A D, [\cdot, \cdot]_*, d)$, are equivalent notions. This parallels the well known fact that Lie-Rinehart structures on (A, L) are in bijective correspondence with differential graded R -algebra structures on $\text{Alt}_A(L, A)$.

Let (A, L', L'') be an almost twilled Lie-Rinehart algebra, having L' and L'' finitely generated and projective as A -modules. The (A, L') -module structure on L'' induces an (A, L') -module on the dual L''^* which, in turn, L''^* being viewed as an abelian Lie algebra and hence abelian (R, A) -Lie algebra, gives rise to the semi direct product (R, A) -Lie algebra $L' \ltimes L''^*$. Likewise the (A, L'') -module structure on L' determines the corresponding semi direct product (R, A) -Lie algebra $L'' \ltimes L'^*$. Plainly $L = L' \ltimes L''^*$ and $D = L'' \ltimes L'^*$ are in duality. Consider the obvious adjointness isomorphisms

$$(4.1.1) \quad \text{Alt}_A(L'', \Lambda_A L') \rightarrow \text{Alt}_A(L'' \ltimes L'^*, A) = \text{Alt}_A(D, A)$$

and

$$(4.1.2) \quad \Lambda_A L = \Lambda_A(L' \ltimes L''^*) \rightarrow \text{Alt}_A(L'', \Lambda_A L')$$

of bigraded A -algebras; these isomorphisms are independent of the Lie-Rinehart semi direct product constructions and instead of $L' \ltimes L''^*$ and $L'' \ltimes L'^*$, we could as well have written $L' \oplus L''^*$ and $L'' \oplus L'^*$, respectively. However, incorporating these semi direct product structures, we see that, under (4.1.1), the Lie-Rinehart differential d'' on $\text{Alt}_A(L'', \Lambda_A L')$ passes to the Lie-Rinehart differential d_* on $\text{Alt}_A(D, A)$ and that under (4.1.2) the (bigraded) Gerstenhaber bracket $[\cdot, \cdot]$ on $\Lambda_A L$ passes to the bigraded Gerstenhaber bracket $[\cdot, \cdot]'$ on $\text{Alt}_A(L'', \Lambda_A L')$. Moreover, by construction, the differentials on both sides of (4.1.1) are derivations with respect to the multiplicative structures.

Theorem 4.1. *For an almost twilled Lie-Rinehart algebra (A, L', L'') having L' and L'' finitely generated and projective as A -modules, $(\text{Alt}_A(L'', \Lambda_A L'), [\cdot, \cdot]', d'')$ is a differential bigraded Gerstenhaber algebra if and only if (A, L, D) is a Lie-Rinehart bialgebra.*

Proof. In fact, the first property spelled out above characterizing (A, L, D) to be a Lie-Rinehart bialgebra is plainly equivalent to $(\text{Alt}_A(L'', \Lambda_A L'), [\cdot, \cdot]', d'')$ being a differential bigraded Gerstenhaber algebra. \square

The following is now immediate, cf. Corollary 4.9 in [18].

Corollary 4.2. *An almost twilled Lie-Rinehart algebra (A, L', L'') having L' and L'' finitely generated and projective as A -modules is a true twilled Lie-Rinehart algebra if and only if $(A, L, D) = (A, L' \ltimes L''^*, L'' \ltimes L'^*)$ is a Lie-Rinehart bialgebra. \square*

This result may be proved directly, i. e. without the intermediate differential bigraded Gerstenhaber algebra in (4.1). The reasoning is formally the same, though. For the special case where L' and L'' arise from Lie algebroids, the statement of Corollary 4.2 may be deduced from what is said in [28].

REMARK 4.3. When A is a field and \mathfrak{g} an ordinary (finite dimensional) Lie algebra, Corollary 4.2 comes down to the statement that, in the terminology of [28], [31],

[33], $(\mathfrak{g}', \mathfrak{g}'')$ (with the requisite additional structure) constitutes a matched pair of Lie algebras (which now amounts to $(\mathfrak{g}', \mathfrak{g}'')$ being a Lie bialgebra) if and only if, with the obvious structure, $(\mathfrak{g}' \ltimes \mathfrak{g}''^*, \mathfrak{g}'' \ltimes \mathfrak{g}'^*)$ is a Lie bialgebra. This fact was known to S. Zakrzewski [43]. It has been spelled out explicitly as Proposition 1 in [37].

References

1. S. Barannikov and M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, Internat. Res. Notices **4** (1998), 201–215, [alg-geom/9710032](#).
2. I. A. Batalin and G. S. Vilkovisky, *Quantization of gauge theories with linearly dependent generators*, Phys. Rev. **D 28** (1983), 2567–2582.
3. I. A. Batalin and G. S. Vilkovisky, *Closure of the gauge algebra, generalized Lie equations and Feynman rules*, Nucl. Phys. B **234** (1984), 106–124.
4. I. A. Batalin and G. S. Vilkovisky, *Existence theorem for gauge algebra*, Jour. Math. Phys. **26** (1985), 172–184.
7. A. Cannas de Silva, K. Hartshorn, A. Weinstein, *Lectures on Geometric Models for Noncommutative Algebras*, U of California at Berkeley, June 15, 1998.
8. C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85–124.
9. M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. **78** (1963), 267–288.
10. M. Gerstenhaber and S. D. Schack, *Algebras, bialgebras, quantum groups and algebraic deformations*, In: Deformation theory and quantum groups with applications to mathematical physics, M. Gerstenhaber and J. Stasheff, eds., Cont. Math. **134** (1992), AMS, Providence, 51–92.
11. E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. in Math. Phys. **195** (1994), 265–285.
12. J. Herz, *Pseudo-algèbres de Lie*, C. R. Acad. Sci. Paris **236** (1953), 1935–1937.
13. J. Huebschmann, *Poisson cohomology and quantization*, J. für die Reine und Angew. Math. **408** (1990), 57–113.
14. J. Huebschmann, *Duality for Lie-Rinehart algebras and the modular class*, Journal reine angew. Math. **510** (1999), 103–159, [dg-ga/9702008](#).
15. J. Huebschmann, *Lie-Rinehart algebras, Gerstenhaber algebras, and Batalin-Vilkovisky algebras*, Annales de l’Institut Fourier **48** (1998), 425–440.
16. J. Huebschmann, *Extensions of Lie-Rinehart algebras and the Chern-Weil construction*, in: Festschrift in honor of J. Stasheff’s 60th birthday, Cont. Math. **227** (1999), Amer. Math. Soc., Providence R. I., 145–176, [dg-ga/9706002](#).
17. J. Huebschmann, *Crossed products and twilled Lie-Rinehart algebras*, in preparation.
18. J. Huebschmann, *Twilled Lie-Rinehart algebras and differential Batalin-Vilkovisky algebras*, [math.DG/9811069](#).
19. J. Huebschmann, *The modular class and master equation for Lie-Rinehart bialgebras*, in preparation.
20. J. Huebschmann and J. D. Stasheff, *Formal solution of the master equation via HPT and deformation theory*, Forum mathematicum **14** (2002), 847–868, [math.AG/9906036](#).

21. Y. Kosmann-Schwarzbach, *Exact Gerstenhaber algebras and Lie bialgebroids*, Acta Applicandae Mathematicae **41** (1995), 153–165.
22. Y. Kosmann-Schwarzbach, *From Poisson algebras to Gerstenhaber algebras*, Annales de l’Institut Fourier **46** (1996), 1243–1274.
23. Y. Kosmann-Schwarzbach, *The Lie bialgebroid of a Poisson-Nijenhuis manifold*, Letters in Math. Physics **38** (1996), 421–428.
24. Y. Kosmann-Schwarzbach and F. Magri, *Poisson-Lie groups and complete integrability. I. Drinfeld bigebras, dual extensions and their canonical representations*, Annales Inst. H. Poincaré Série A (Physique théorique) **49** (1988), 433–460.
25. J. L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, in E. Cartan et les Mathématiciens d’aujourd’hui, Lyon, 25–29 Juin, 1984, Astérisque, **hors-série**, (1985), 251–271.
26. B. H. Lian and G. J. Zuckerman, *New perspectives on the BRST-algebraic structure of string theory*, Comm. in Math. Phys. **154** (1993), 613–646.
27. J.-H. Lu and A. Weinstein, *Poisson Lie groups, dressing transformations, and Bruhat decompositions*, J. of Diff. Geom. **31** (1990), 501–526.
28. K. Mackenzie, *Double Lie algebroids and the double of a Lie bialgebroid*, math.DG/9808081.
29. K. C. Mackenzie and P. Xu, *Lie bialgebroids and Poisson groupoids*, Duke Math. J. **73** (1994), 415–452.
30. S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften No. 114, Springer, Berlin · Göttingen · Heidelberg, 1963.
31. S. Majid, *Matched pairs of Lie groups associated to solutions of the Yang-Baxter equation*, Pac. J. of Math. **141** (1990), 311–332.
32. Yu. I. Manin, *Three constructions of Frobenius manifolds*, math.QA/9801006.
33. T. Mokri, *Matched pairs of Lie algebroids*, Glasgow Math. J. **39** (1997), 167–181.
34. R. S. Palais, *The cohomology of Lie rings*, Amer. Math. Soc., Providence, R. I., Proc. Symp. Pure Math. **III** (1961), 130–137.
35. G. Rinehart, *Differential forms for general commutative algebras*, Trans. Amer. Math. Soc. **108** (1963), 195–222.
36. V. Schechtman, *Remarks on formal deformations and Batalin-Vilkovisky algebras*, math.AG/9802006.
37. P. Stachura, *Double Lie algebras and Manin triples*, q-alg/9712040.
38. J. D. Stasheff, *Deformation theory and the Batalin-Vilkovisky master equation*, in: Deformation Theory and Symplectic Geometry, Proceedings of the Ascona meeting, June 1996, D. Sternheimer, J. Rawnsley, S. Gutt, eds., Mathematical Physics Studies, Vol. 20 (1997), Kluwer Academic Publishers, Dordrecht/Boston/London, 271–284.
39. G. Tian, *A note on Kaehler manifolds with $c_1 = 0$* , preprint.
40. A. N. Todorov, *The Weil-Petersson geometry of the moduli space of $\mathfrak{su}(n \geq 3)$ (Calabi-Yau) manifolds, I*, Comm. Math. Phys. **126** (1989), 325–346.
41. E. Witten, *Mirror manifolds and topological field theory*, in: Essays on mirror manifolds, S. T. Yau, ed. (1992), International Press Co., Hong Kong, 230–310.
42. P. Xu, *Gerstenhaber algebras and BV-algebras in Poisson geometry*, preprint, 1997.

43. S. Zakrzewski, *Poisson structures on the Poincaré groups*, Comm. Math. Phys. **185** (1997), 285–311.